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Model Predictive Control of Nonlinear Parameter Varying Systems via Receding Horizon Control Lyapunov Functions *

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Abstract

The problem of rendering the origin an asymptotically stable equilibrium point of a nonlinear system while, at the same time, optimizing some measure of performance has been the object of much attention in the past few years. In contrast to the case of linear systems where several optimal synthesis techniques (such as $\mathcal{H}_\infty$, $\mathcal{H}_2$ and ℓ^1) are well established, their nonlinear counterparts are just starting to emerge. Moreover, in most cases these tools lead to partial differential equations that are difficult to solve. In this chapter we propose a suboptimal regulator for nonlinear parameter varying, control affine systems based upon the combination of model predictive and control Lyapunov function techniques. The main result of the chapter shows that this controller is nearly optimal provided that a certain finite horizon problem can be solved on–line. Additional results include (a) sufficient conditions guaranteeing closed loop stability even in cases where there is not enough computational power available to solve this optimization on–line; and (b) an analysis of the suboptimality level of the proposed method.

1   Introduction

A large number of control problems involve designing a controller capable of rendering some point an asymptotically stable equilibrium point of a given time invariant system while, simultaneously, optimizing some performance index. In the case of linear dynamics this problem has been thoroughly explored during the past decade, leading to powerful formalisms such as μ-synthesis and ℓ^1 optimal control theory that have been successfully employed to solve some hard practical problems.

In the case of nonlinear dynamics, popular design techniques include Jacobian linearization (JL) [14], feedback linearization (FL) [14], the use of control Lyapunov functions (CLF)[1, 24] and recursive backstepping [14]. While these methods provide powerful tools for designing stabilizing controllers, performance of the resulting closed loop systems can vary widely, as we illustrate in the sequel with the problem of controlling a thrust vectored aircraft. A simplified planar model of the system is shown in Figure 1, with the corresponding dynamics given by (see [30, 7] for details):

$$
\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -g\sin\theta \\ g(\cos\theta - 1) \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\cos\theta}{m} & -\frac{\sin\theta}{m} \\ \frac{\sin\theta}{m} & \frac{\cos\theta}{m} \\ \frac{r}{J} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\tag{1}
$$

where x, y and θ denote horizontal, vertical and angular position respectively and where u_1 and $u_2 = u_2 + mg$[1] are the control inputs.
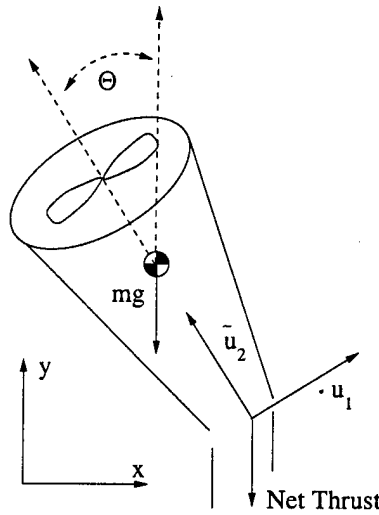

Figure 1: Simplified model of a thrust vectored aircraft

Assume that the goal is to drive the system to the origin, while minimizing a performance index of the form

$$
J(x_o, u) = \int_0^\infty \left[\xi' Q \xi + u' R u \right] dt,
\tag{2}
$$

$$
\xi(0) = \begin{bmatrix} 0 & 0 & 0 & 12.5 & 0 & 0 \end{bmatrix}
\tag{3}
$$

$$
Q = diag\begin{bmatrix} 5 & 5 & 1 & 1 & 1 & 5 \end{bmatrix}, R = I_{2\times2}
\tag{4}
$$

corresponding to the following choice of state variables: $\xi = \begin{bmatrix} x & y & \theta & \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}$.

---

[1]Following [30] the control u_2 has been shifted to compensate for gravity.

method	cost
exact	1115
CLF [7]	2.53×10^4
JL + LQR [7]	1.1×10^5

Table 1: Comparison of different methods for the thrust vectored aircraft example

Table 1 compares the performance achieved by several commonly used nonlinear control design methods. As shown there, in this case the performance of the CLF and JL controllers is worse, by an order of magnitude, than the optimal cost (obtained by offline optimization using a conjugate gradient algorithm). Indeed, a recent workshop on nonlinear control [7] has shown that while the methods mentioned aboved can recover the optimal control under certain conditions, in general there are no guarantees on the performance of the resulting system.

As an alternative, nonlinear counterparts of $\mathcal{H}_\infty$ [3, 11] and ℓ^1 [15] have recently started to emerge. While these theories can guarantee optimality (at least in a certain sense), from a practical standpoint they suffer from the fact that they lead to Hamilton–Jacobi–Isaacs type partial-differential equations that are hard to solve, except in some restrictive, low–dimensional cases.

Given these practical difficulties, during the past few years there has been an increased interest in extending receding horizon (RH) techniques to nonlinear plants. These techniques are appealing since they allow for explicitly handling constraints and guarantee optimality in some sense. Moreover, since the optimization is carried only along the present trajectory of the system (i.e "locally") the resulting computational complexity is far less than that associated with finding the true global optimal control (a task that entails solving a Hamilton–Jacobi type equation). However, in contrast with the linear case (where global stability has been established [27, 23, 21]), for nonlinear plants only local stability results are available [17]. Several modified nonlinear RH formulations addressing this problem have been proposed, mostly based on the use of additional constraints or a terminal penalty. For instance, [18] uses dual control, where an explicit optimization is used to drive the system to a neighborhood of the origin, where a locally stabilizing linear control law is used. Magni [16] uses a terminal penalty obtained assuming that a linear control law will be used after the optimization horizon T. Finally, [20] achieves stability by enforcing an additional state constraint. However, while these approaches guarantee closed–loop stability, they may do so at the expense of performance.

In this chapter we propose an alternative controller for suboptimal regulation of nonlinear, parameter varying, control affine systems, based upon the combination of receding horizon and control Lyapunov functions ideas. This approach follows in the spirit of a similar controller successfully used in the case of constrained linear systems [25, 26, 27], where receding horizon was used to drive the system to an invariant neighborhood of the origin where a stabilizing controller is available. In the first part of the chapter we show that combining these ideas with a suitable finite–horizon approximation of the performance index, leads to globally stabilizing, nearly optimal controllers, provided that enough computational power is available to solve on–line an optimization problem. In the second part of the chapter we show how to modify these controllers to guarantee global stability in the face of computational time constraints. Additional results include an analysis of the suboptimality of the proposed method and show that if an approximate solution to the problem is known in a set containing the origin, then our controller yields an extension of this solution with the same suboptimality level. Finally, we show that in the limit as the optimization horizon $T \to 0$, the

method reduces to the well known inverse optimal controller of Freeman and Kokotovic [8].

2 Preliminaries

2.1 Notation and Definitions

In the sequel we consider the following class of control–affine nonlinear parameter varying (NLPV) systems:

$$\dot{x} = f[x, \rho(t)] + g[x, \rho(t)]u \tag{5}$$

where $x \in R^n$ and $u \in R^m$ represent the state and control variables, the vector fields $f(.,.)$ and $g(.,.)$ are known C^1 functions, and where $\rho \in R^{n_\rho}$ denotes a vector of time–varying parameters, unknown a priori, but available to the controller in real time. Further, we will assume that at all times $\rho(t) \in \mathcal{P} \subset R^{n_\rho}$, where $\mathcal{P}$ is a given compact set, and that the set of admissible parameter trajectories is given by:

$$\mathcal{F}_v \doteq \left\{ \rho \in C^1(R, R^{n_\rho}) : \rho(t) \in \mathcal{P}, \underline{v}_i \leq \dot{\rho}_i(t) \leq \bar{v}_i, i = 1, 2, \ldots n_\rho, \forall\, t \in R_+ \right\} \tag{6}$$

where $\underline{v}_i$ and $\bar{v}_i$ are given numbers.

Definition 1 *A positive definite C^1 function $V : R^n \times R \rightarrow R_+$ is a Control Lyapunov function (CLF) for the system (5) if it is radially unbounded in x and*

$$\inf_u \left[L_f V(x, \rho) + L_g V(x, \rho)u + \frac{\partial V}{\partial \rho}\dot{\rho} \right] \leq -\sigma(x) < 0, \qquad \forall\, x \neq 0 \text{ and all } \rho \in \mathcal{F}_v \tag{7}$$

where $\sigma(.)$ is a positive definite function, and where $L_h V(x, \rho) = \frac{\partial V}{\partial x}h(x, \rho)$ denotes the Lie derivative of V along h.

2.2 The Quadratic Regulator Problem for NLPV Systems

Consider the NLPV system (5). In the sequel we consider the following problem: Given an initial condition x_o and an initial value of the parameter ρ_o, find a parameter dependent state–feedback control law $u[x(t), \rho(t)]$ that minimizes the following performance index:

$$J(x_o, \rho_o, u) = \sup_{\rho \in \mathcal{F}_v, \rho(0) = \rho_o} \int_0^\infty \left[x'Q(x, \rho)x + u'R(x, \rho)u \right] dt, \; x(0) = x_o \tag{8}$$

where $Q(.)$ and $R(.)$ are C^1, positive definite matrices[2]. In the sequel, for simplicity, the explicit dependence of matrices on x and ρ will be omitted, when it is clear from the context.

---

[2]This condition can be relaxed to $Q(x, \rho) \geq 0$

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By using Pontryagin's principle it can be shown that solving this problem is equivalent to solving the following Hamilton–Jacobi–Bellman type partial differential equation:

$$0 = \frac{\partial V}{\partial x} f - \frac{1}{4} \frac{\partial V}{\partial x} g R^{-1} g' \frac{\partial V'}{\partial x} + \frac{\partial V}{\partial t} + x' Q x + \max_{\underline{v}_i \leq v_i \leq \overline{v}_i} \sum_{i=1}^{n_\rho} \frac{\partial V}{\partial \rho_i} v_i \qquad \text{subject to } V(0,\rho)=0 \qquad (9)$$

If this equation admits a C^1 nonnegative solution V, then the optimal control is given by $u(x,\rho) = -\frac{1}{2} R^{-1} g' \frac{\partial V'}{\partial x}$ and $V(x,\rho)$ is the corresponding optimal cost (or storage function), i.e.

$$V(x,\rho) = \min_u \sup_{\rho \in \mathcal{F}_v} \int_0^\infty \left(x' Q x + u' R u \right) dt$$

3 An Equivalent Finite Horizon Regulation Problem

Unfortunately, the complexity of equation (9) prevents its solution, except in some very simple, low dimensional cases. To solve this difficulty, motivated by the work in [25, 28], in this section we introduce a finite horizon approximation to the nonlinear regulation problem.

Lemma 1 *Consider a compact set S containing the origin in its interior and assume that the optimal storage function $V(x,\rho)$ is known for all $x \in S, \rho \in \mathcal{P}$. Let $v = \min_{x \in \partial S} \min_{\rho \in \mathcal{P}} V(x,\rho)$ where ∂S denotes the boundary of S. Finally, define the set $S_v = \left\{ x : \sup_{\rho \in \mathcal{P}} V(x,\rho) \leq v \right\}$. Consider the following two optimization problems:*

$$\min_u \sup_{\rho \in \mathcal{F}_v, \rho(t)=\rho_o} \left\{ J(x_o, u, \rho_o) = \int_t^\infty [x' Q x + u' R u] d\tau \right\} \qquad (10)$$

$$\min_u \sup_{\rho \in \mathcal{F}_v, \rho(t)=\rho_o} \left\{ J_T(x_o, u, \rho_o) = \int_t^T [x' Q x + u' R u] d\tau + V[x(T), \rho(T)] \right\} \qquad (11)$$

subject to (5) with $x(t) = x_o$. Then an optimal solution of problem (11) is also optimal for (10) in the interval $[t, T]$ provided that $x(T) \in S_v$.

Proof: If $x_o \in S_v$ the proof follows immediately from the facts that, for all admissible parameter trajectories, S_v is positively invariant and that $V(x,\rho)$ is the optimal return function there. If $x_o \notin S_v$, consider the following free terminal time problem:

$$J^o(x,\rho,t) = \min_u \sup_{\rho \in \mathcal{F}_v} \left\{ V[x(t_f),\rho(t_f)] + \int_t^{t_f} [x' Q x + u' R u] dt \right\}$$
$$\text{subject to:} \qquad (12)$$
$$x(t_f) \in S_v$$

Let x^o, u^o, ρ^o denote the optimal trajectory. It can be easily seen that the optimal return function satisfies:

$$0 = \frac{\partial J}{\partial t} + \frac{\partial J}{\partial x} f(x,\rho) - \frac{1}{4} \frac{\partial J}{\partial x} g R^{-1} g' \frac{\partial J'}{\partial x} + x' Q x + \max_{\underline{v}_i \leq v_i \leq \overline{v}_i} \sum_{i=1}^{n_\rho} \frac{\partial J}{\partial \rho_i} v_i \qquad (13)$$

with boundary condition $J(x,\rho,t) = V(x,\rho)$ for $x \in S_v$. Clearly this equation admits as solution $J(x,\rho,t) \equiv V(x,\rho)$. Thus problems (10) and (12) are equivalent. To establish the claim we will show that an optimal

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solution u^o of (11) is also optimal for (12) (and thus (10)), provided that $x^o(T) \in S_v$. To this effect note that the Euler–Lagrange optimality conditions for problems (11) and (12) are identical, except for the additional transversality condition $H[u^o, x^o(t_f), \lambda^o(t_f), \mu^o(t_f)] = 0$ that appears in the latter, where $\lambda(t)$ and $\mu(t)$ denote the co–states associated with the states and parameters, respectively. The boundary conditions for these co–states in problem (11) are given by

$$\lambda^o(T) = \frac{\partial V}{\partial x}\Big|_{x(T), \rho(T)}, \ \mu^o(T) = \frac{\partial V}{\partial \rho}\Big|_{x(T), \rho(T)} \tag{14}$$

Since $x(T) \in S_v$ it follows that $x^o(T), u^o(T), \lambda^o(T), \mu^o(T)$ satisfy the HJB equation (9), or equivalently $H[u^o, x^o(T), \lambda^o(T), \mu^o(T)] = 0$. Thus, an optimal solution of (11) is also optimal for (12).

$\square$

This lemma shows that if a solution to the HJB equation (9) is known in a neighborhood of the origin, then it can be extended via an explicit finite horizon optimization, well suited for an on–line implementation. This suggests the following RH type control law:

Algorithm 1

0.- Data: The region S_v, the function $V(x, \rho)$ for all $x \in S_v$, a sampling interval δT.

1.- If $x(t) \in S_v$, $u = -\frac{1}{2} R^{-1} g(x, \rho)' \frac{\partial V(x, \rho)'}{\partial x}$

2.- If $x(t) \notin S_v$ then solve a sequence of optimization problems of the form (11) with increasing values of T until a solution such that $x(T) \in S_v$ is found. Use the corresponding control law $u(t)$ in the interval $[t, t + \delta T]$.

From the results above it is clear that the resulting control law is globally optimal and thus globally stabilizing. However, the computational complexity associated with finding $V(x, \rho)$ (even only in the region S_v) may preclude the use of this control law in many practical cases. Thus, it is of interest to consider a control law where an approximation $\Psi(x, \rho)$ (rather than $V(x, \rho)$)is used. To this effect consider a compact set S containing the origin in its interior and let $\Psi : S \times \mathcal{P} \to R_+$, $\Psi \in C^1(R^{n_x} \times R^{n_\rho}, R)$ be a Control Lyapunov Function for system (5). Finally, let $c = \min_{x \in \partial S} \min_{\rho \in \mathcal{P}} \Psi(x, \rho)$ and define the set $S_\Psi \subseteq S = \{x : \max_{\rho \in \mathcal{P}} \Psi(x, \rho) \leq c\}$. Consider the modified control law:

Algorithm 2

0.- Data: a CLF $\Psi(x, \rho)$, the region S_Ψ, a sampling interval δT, a positive definite function $\sigma(.)$.

1.- If $x(t) \in S_\Psi$, $u_\Psi(x, \rho) = \underset{u}{\text{argmin}} \ \underset{\underline{v}_i \leq v_i \leq \bar{v}_i}{\sup} \left\{ \|u\| : \frac{\partial \Psi}{\partial x}[f(x, \rho) + g(x, \rho)u] + \Sigma_i \frac{\partial \Psi}{\partial \rho_i} v_i \leq -\sigma(x) < 0 \right\}$

2.- If $x \notin S_\Psi$ then consider an increasing sequence T_i. Let

$$u_{T_i}^* = \underset{u}{\text{argmin}} \left\{ \underset{\rho \in \mathcal{F}_v}{\sup} \left[\int_t^{T_i} (x'Qx + u'Ru) \, d\tau + \Psi[x(T_i), \rho(T_i)] \right] \right\}$$

Denote by $x^*(.)$ the corresponding optimal trajectory and define:[3]*:*

$$J(x_o, \rho_o, T_i) = \sup_{\rho \in \mathcal{F}_v} \left[\int_t^{T_i} \left(x^{*\prime} Q x^* + u^{*\prime} R u^* \right) d\tau + \Psi \left[x^*(T_i), \rho(T_i) \right] \right]$$

$$T(x, \rho_o) = \operatorname{argmin}_T \left\{ J(x, \rho_o, T_i) : x^*(T_i) \in S_\Psi \text{ for all } \rho(.) \in \mathcal{F}_v, \rho(t) = \rho_o \right\}$$

(15)

Then $u_\Psi(x, \rho) \doteq u^*_{T(x)}(\tau), \tau \in [t, t + \delta T]$.

Theorem 1 *Assume that $Q(x, \rho) \geq \sigma_m I > 0$. Then the control law u_Ψ generated by Algorithm 2 renders the origin a globally asymptotically stable equilibrium point of (5).*

<u>Proof:</u> Consider first an initial condition $x_o \notin S_\Psi$, an initial value of the parameter ρ_o, and the corresponding optimal control law $u^*(.)$ and trajectory $x^*(.)$. Let

$$J(x_o, \rho_o) = \sup_{\rho \in \mathcal{F}_v : \rho(t) = \rho_o} \left\{ \int_t^{T(x_o, \rho_o)} \left(x^{*\prime} Q x^* + u^{*\prime} R u^* \right) d\tau + \Psi \left\{ x^*[T(x_o, \rho_o)], \rho[T(x_o, \rho_o)] \right\} \right\}$$

and define $x_1 \doteq x^*(t + dt)$ (with $dt > 0$ small enough so that $x^*(t + dt) \notin S_\Psi$) and $\rho_1 \doteq \rho(t + dt)$, where $\rho(.)$ denotes any admissible parameter trajectory starting at ρ_o. Since $x^*(\tau), \tau \in [t + dt, T]$ is also a feasible trajectory starting from x_1 we have that:

$$
\begin{aligned}
J(x_1, \rho_1) &= \inf_u \sup_{\rho \in \mathcal{F}_v : \rho(t + dt) = \rho_1} \left\{ \int_{t+dt}^{T(x_1, \rho_1)} (x' Q x + u' R u) d\tau + \Psi \left\{ x[T(x_1, \rho_1)], \rho[T(x_1, \rho_1)] \right\} \right\} \\
&\leq \sup_{\rho \in \mathcal{F}_v : \rho(t + dt) = \rho_1} \int_{t+dt}^{T(x_o, \rho_o)} \left(x^{*\prime} Q x^* + u^{*\prime} R u^* \right) d\tau + \Psi \left\{ x^*[T(x_o, \rho_o)], \rho[T(x_o, \rho_o)] \right\} \\
&\leq \sup_{\rho \in \mathcal{F}_v : \rho(t) = \rho_o} \left\{ \int_t^{T(x_o, \rho_o)} \left(x^{*\prime} Q x^* + u^{*\prime} R u^* \right) d\tau + \Psi \left\{ x^*[T(x_o, \rho_o)], \rho[T(x_o, \rho_o)] \right\} \right\} \\
&\quad - \int_t^{t+dt} (x^{*\prime} Q x^* + u^{*\prime} R u^*) d\tau = J(x_o) - \int_t^{t+dt} (x^{*\prime} Q x^* + u^{*\prime} R u^*) d\tau
\end{aligned}
$$

(16)

Thus, for any admissible parameter trajectory,

$$\dot{J} = \lim_{dt \to 0} \frac{J[x(t + dt), \rho(t + dt)] - J[x(t), \rho(t)]}{dt} \leq -x(t)' Q x(t) \leq -\sigma_m \cdot \|x\| < 0$$

Since $J(x, \rho) > 0$ and $\dot{J}(x, \rho) < 0$ for all $x \notin S_\Psi$, it follows that all trajectories reach the set S_Ψ in finite time. Asymptotic stability now follows from the facts that S_Ψ is invariant with respect to u_Ψ (i.e. trajectories starting in the set never leave it) and that $\Psi(x, \rho)$ is a CLF there.

□

4 A Modified Receding Horizon Controller

In the last section we outlined a receding horizon type law, that under certain conditions globally stabilizes system (5). While most of these conditions are rather mild (essentially equivalent to the existence of a CLF),

---

[3]From Lasalle–Yoshizawa's Theorem [14] we have that $\lim_{T \to \infty} x' Q x = 0$. Hence $T(x_o, \rho_o)$ is finite.

the requirement that T should be large enough so that $x(T) \in S_\Psi$ could pose a problem, specially in cases where the system has fast dynamics. Thus, it is of interest to consider the following modified control law where both an approximation $\Psi(x,\rho)$ (rather than $V(x,\rho)$) and a fixed horizon T are used:

Algorithm 3

0.- *Data: a CLF $\Psi(x)$, an invariant region S_Ψ such that $0 \in int\{S_\Psi\}$, a horizon T.*

1.- *If $x(t) \in S_\Psi$, $u_\Psi(x,\rho) = \underset{u}{argmin} \left\{ \|u\|: \underset{\underline{v}_i \le v_i \le \bar{v}_i}{\sup} L_f\Psi + L_g\Psi u + \sum_i \frac{\partial \Psi}{\partial \rho_i} v_i \le -\sigma(x) < 0 \right\}$*

2.- *If $x(t) \notin S_\Psi$ then $u_\Psi(x,\rho) = u(t)$ where $u(\lambda)$, $\lambda \in [t, t+T]$ is given by:*

$$u = \underset{u}{argmin} \underset{\rho \in \mathcal{F}_v}{\sup} \left\{ \int_t^{T+t} (x'Qx + u'Ru)\, d\tau + \Psi[x(T+t), \rho(T+t)] \right\} \tag{17}$$

subject to:

$$0 \ge x(t+T)'Qx(t+T) + \underset{v}{\min} \underset{\rho \in \mathcal{F}_v}{\sup} \left\{ v'Rv + \frac{d\Psi}{dt}\Big|_{x(t+T)} \right\} - u'(t)Ru(t) \tag{18}$$

Note that inside the region S_Ψ, the control action is a generalization to the NLPV case of the pointwise–minimum norm controller proposed in [8]. Proceeding as in [10] it can be shown that if the CLF Ψ has the same level sets as the optimal value function V inside S_Ψ, then u_Ψ is indeed the optimal solution for the original problem. Thus, the algorithm allows for "switching–off" the on–line optimization when inside a region where the optimal storage function (or a good approximation) is known.

Theorem 2 *The control law u_Ψ generated by Algorithm 3 has the following properties:*

1. *It renders the origin a globally asymptotically stable equilibrium point of (5).*

2. *It coincides with the globally optimal control law when $\Psi(x,\rho) = V(x,\rho)$, where $V(x,\rho)$ denotes the optimal storage function obtained by solving (9).*

Proof: To prove stability, proceeding as in Theorem 1, consider first an initial condition $x_o \notin S_\Psi$. Denote by u^*, x^* the optimal control and associated trajectory respectively. Then

$$
\begin{aligned}
J[x(t+dt), \rho(t+dt)] &= \underset{u}{\min} \underset{\rho}{\sup} \left\{ \int_{t+dt}^{T+t+dt} (x'Qx + u'Ru)\, dt + \Psi[x(T+t+dt), \rho(T+t+dt)] \right\} \\[2mm]
&\le \underset{\rho}{\sup} \int_{t+dt}^{T+t} (x^{*\prime}Qx^* + u^{*\prime}Ru^*)\, dt + \Psi[x^*(T+t), \rho(T+t)] \\
&\quad + \underset{v}{\min} \underset{\rho}{\sup} \left\{ x^{*\prime}(T+t)Qx^*(T+t) + v'Rv + \dot{\Psi}[x^*(T+t), \rho] \right\} dt \\[2mm]
&= J[x(t), \rho(t)] - \left[x^*(t)'Qx^*(t) + u^{*\prime}(t)Ru^*(t) \right] dt \\
&\quad + \underset{v}{\min} \underset{\rho}{\sup} \left\{ x^{*\prime}(T+t)Qx^*(T+t) + v'Rv + \dot{\Psi}[x^*(T+t), \rho] \right\} dt
\end{aligned} \tag{19}
$$

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Therefore, if (18) holds then we have that, for all admissible parameter trajectories:

$$\dot{J} = \lim_{dt \to 0} \frac{J[x(t+dt), \rho(t+dt)] - J[x(t), \rho(t)]}{dt} \leq -\sigma_m \|x\|_2^2 \tag{20}$$

where σ_m denotes the minimum singular value of Q. Hence the trajectories starting outside S_Ψ reach this set in a finite time. As in the proof of Theorem 1, once there, asymptotic stability is guaranteed by the parameter dependent control Lyapunov function $\Psi(x, \rho)$.

To prove item 2.- note that if $\Psi(x, \rho) = V(x, \rho)$ then from the Hamilton Jacobi equation (9) we have that,

$$x(t+T)'Qx(t+T) + \min_v \sup_\rho \left\{ v'Rv + \dot{\Psi} \Big|_{(t+T)} \right\} = 0 \tag{21}$$

Thus the constraint (18) is redundant and the proof follows immediately from Lemma 1.

$\square$

Finally, before closing this section we consider a modified control law that takes into account the sample and hold nature of receding horizon implementations.

Algorithm 4

0.- Data: a CLF $\Psi(x)$, an invariant region S_Ψ such that $0 \in int\{S_\Psi\}$, a horizon T, a sampling interval δT.

1.- If $x(t) \in S_\Psi$, $u_\Psi(x, \rho) = \operatorname{argmin}_u \left\{ \|u\|: \sup_{\underline{v}_i \leq v_i \leq \overline{v}_i} L_f \Psi + L_g \Psi u + \sum_i \frac{\partial \Psi}{\partial \rho_i} v_i \leq -\sigma(x) < 0 \right\}$

2.- If $x(t) \notin S_\Psi$ then $u_\Psi(x, \rho) = u^*(\tau)$, $\tau \in [t, t+\delta T]$, where $u^*(.)$ is given by:

$$u^* = \operatorname{argmin}_u \sup_{\rho \in \mathcal{F}_v} \left\{ \int_t^{T+t} (x'Qx + u'Ru)\, d\tau + \Psi[x(T+t), \rho(T+t)] \right\} \tag{22}$$

subject to:

$$0 \geq x(t+T+\tau)'Qx(t+T+\tau) + \min_u \sup_{\rho \in \mathcal{F}_v} \left\{ u'(t+T+\tau)Ru(t+T+\tau) + \dot{\Psi} \Big|_{(t+T+\tau)} \right\} - u'(t)Ru(t)$$
$$\text{for all } 0 \leq \tau \leq \delta T \tag{23}$$

Lemma 2 *The control law u_s renders the origin a globally stable equilibrium point of (5). Moreover, it coincides with the globally optimal control law when $\Psi(x, \rho) = V(x, \rho)$ and $\delta T \to 0$.*

Proof: The proof, omitted for space reasons, follows along the lines of the proof of Theorem 2.

5 Selecting suitable CLFs

In principle, any of the methods available in the literature for finding CLFs such as feedback linearization and backstepping (see for instance [9]) can be used to find the function $\Psi(.)$. Alternatively, if a stabilizing linear static feedback control law $u_k = Kx$ is known, then, following [16] a suitable CLF is given by:

$$\Psi(x,\rho) = \sup_{\rho \in \mathcal{F}_v} \int_t^\infty x'(Q + K'RK)x\, dt$$

However, as we show next, in some cases of practical interest, specific families of CLFs are readily available that reduce the degree of suboptimality incurred by the algorithm.

5.1 Autonomous systems

Consider the case of autonomous nonlinear systems, i.e. where $f = f(x)$ and $g = g(x)$ in (5). As we show in the sequel, in this case the suboptimality level incurred by the proposed algorithm is roughly similar to the difference between the CLF $\Psi(.,.)$ and the actual value function $V(.,.)$.

Theorem 3 *Let $\Psi: R^n \to R_+$ be a positive definite, radially unbounded function and consider the following optimization problem:*

$$J_\Psi(x,t) = \min_u \int_t^T \left(x'Qx + u'Ru\right)d\tau + \Psi[x(T)] \tag{24}$$

subject to (5). Then

$$J_\Psi(x,t) - V[x(t)] = \Psi[x(T)] - V[x(T)] + O\left[\|\frac{\partial e}{\partial x}|_{x(T)}\|^2\right] + O(dt^2) \tag{25}$$

where $e(x,t) \doteq J_\Psi(x,t) - V(x)$ denotes the approximation error.

<u>Proof:</u> By considering the Hamilton Jacobi equations for J_Ψ and V it can be easily shown that $e(t,x)$ satisfies the following equation:

$$0 = \frac{\partial e}{\partial t} + \frac{\partial e}{\partial x}\left(f - \frac{1}{2}gR^{-1}g'\frac{\partial J_\Psi}{\partial x}\right) + \frac{1}{4}\frac{\partial e}{\partial x}gR^{-1}g'\frac{\partial e'}{\partial x} \tag{26}$$

By exploiting the fact that the optimal control law for (24) is given by

$$u_\Psi = -\frac{1}{2}R^{-1}g'\frac{\partial J_\Psi}{\partial x} \tag{27}$$

equation (26) can be rewritten as:

$$
\begin{aligned}
0 &= \tfrac{\partial e}{\partial t} + \tfrac{\partial e}{\partial x}(f - gu_\Psi) + \tfrac{1}{4}\tfrac{\partial e}{\partial x}gR^{-1}g'\tfrac{\partial e'}{\partial x} \\
&= \tfrac{\partial e}{\partial t} + \tfrac{\partial e}{\partial x}\dot{x} + \tfrac{1}{4}\tfrac{\partial e}{\partial x}gR^{-1}g'\tfrac{\partial e'}{\partial x} \\
&= \dot{e} + \tfrac{1}{4}\tfrac{\partial e}{\partial x}gR^{-1}g'\tfrac{\partial e'}{\partial x}
\end{aligned} \tag{28}
$$

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From this last equation it follows that:

$$\dot{e}(T) = -\frac{1}{4}\frac{\partial e}{\partial x}gR^{-1}g'\frac{\partial e'}{\partial x}\Big|_{x(T)} \tag{29}$$

Expanding $e(t)$ in a Taylor series around $t = T$ yields:

$$
\begin{aligned}
e(t) &= e(T) + \dot{e}(T)dt + O(dt^2) \\
&= e(T) - \frac{1}{4}\frac{\partial e}{\partial x}gR^{-1}g'\frac{\partial e'}{\partial x}\Big|_{x(T)} dt + O(dt^2)
\end{aligned} \tag{30}
$$

$\square$

Corollary 1 *Assume that Ψ is selected so that $\left\| \frac{\partial e[x(T),T]}{\partial x} \right\| \cong 0$. Then $J_\Psi(x,t) - V[x(t)] \cong \Psi[x(T)] - V[x(T)]$ (to the first order in dt) along the trajectories of the system.*

The result above formalizes the intuitively appealing fact that in order to improve performance, the CLF $\Psi(x)$ should be selected "close" to V. Next we briefly discuss an approach to generate such functions. This approach is motivated by the empirically observed success of the State Dependent Riccati Equation (SDRE) method, briefly covered in the Appendix.

From Lemma 4 in the Appendix, it follows that $\Psi(x) = x'P(x)x$, where $P(x)$ denotes the solution to the SDRE, is a CLF in a neighborhood of the origin. Moreover, since the control law u_{sdre} is locally stabilizing, it can be easily shown that there exists T_o (possibly depending on the initial condition) such for all $T > T_o$ the constraints (18) are feasible. It follows that $\Psi(x) = x'P(x)x$ is a suitable choice for the terminal penalty. Moreover from the properties of the SDRE method (see the Appendix) it follows that with this choice, the control law satisfies all the necessary conditions for optimality as $O\left[\|x(t+T)\|^2\right]$. Thus, we will expect that Algorithm 3 using $\Psi(x) = x'P(x)x$ will generate a nearly optimal control law, even when T is relatively small. In section 8 we will show that this is indeed the case with two examples.

5.2 Linear Parameter Varying Systems

Consider now the case where the dynamics (5) are linear in the state, i.e.

$$\dot{x} = A[\rho(t)]x + B[\rho(t)]u$$

This is case of LPV systems, that has been the object of much attention in the past few years [31], as a vehicle to formalize the concept of gain scheduling. In this case it can be shown that a parameter dependent state feedback controller with guaranteed performance can be synthesized by solving the following convex optimization problem (see [2] for details):

$$\min_{X(\rho)>0,Z} \text{Trace} Z \tag{31}$$

subject to:

$$\left[\begin{matrix} -\sum_{i=1}^{n_\rho} \underline{\nu}_i \frac{\partial X(\rho)}{\partial \rho_i} + A(\rho)X(\rho) + X(\rho)A'(\rho) - B(\rho)R^{-1}B'(\rho) & X(\rho) \\ X(\rho) & -Q \end{matrix} \right] < 0$$

$$\left[\begin{matrix} -\sum_{i=1}^{n_\rho} \bar{\nu}_i \frac{\partial X(\rho)}{\partial \rho_i} + A(\rho)X(\rho) + X(\rho)A'(\rho) - B(\rho)R^{-1}B'(\rho) & X(\rho) \\ X(\rho) & -Q \end{matrix} \right] < 0 \tag{32}$$

$$\left[\begin{matrix} Z & I \\ I & X(\rho) \end{matrix} \right] > 0$$

for all $\rho \in \mathcal{P}$. The corresponding control action is given by

$$u = -\frac{1}{2}R^{-1}B'(\rho)X^{-1}(\rho)x \tag{33}$$

and the corresponding cost is bounded by:

$$J(x_o, \rho_o) = \sup_{\rho \in \mathcal{F}_\nu} \int_0^\infty \left(x'Qx + u'Ru \right) dt \le x_o'X^{-1}(\rho_o)x_o \tag{34}$$

While this approach yields a stabilizing controller with guaranteed performance bounds, it is potentially conservative due to the facts that (i) it uses a quadratic parameter dependent Lyapunov function ($x'X^{-1}(\rho)x$), and (ii) allows for all possible combinations of the parameters and their derivatives. In the sequel, we indicate how performance can be improved by exploiting the Receding Horizon ideas presented in this chapter.

Lemma 3 *Consider the case of LPV dynamics. If the terminal penalty Ψ in Algorithm 3 is chosen as $\Psi(x, \rho) = x'X^{-1}(\rho)x$, where X denotes any feasible solution to the set of affine matrix inequalities (AMIs) (32) then the following holds:*

1. *The resulting control law $u_\psi(x, \rho)$ is globally stabilizing for any choice of the horizon T.*

2. *For any admissible parameter trajectory ρ we have that:*

$$J_\psi(x_o, \rho) \doteq \int_0^\infty x'Qx + u'Rudt \le x(0)'X^{-1}(\rho_o)x(0) \tag{35}$$

   i.e. the proposed control law is guaranteed to perform no worse than the AMI based control law (33), in the sense that both have the same worse–case upper bound.

Proof: To establish the first claim, note that from the Euler Lagrange conditions for optimality, it can be easily shown that in this case the constraint (18) is redundant, since it is satisfied by the control that optimizes the performance index (17). Stability follows now from Theorem 2.

To complete the proof, let x_{AMI}, u_{AMI}, x_ψ and u_ψ denote the trajectory and control corresponding to the parameter trajectory ρ, obtained when using the AMI–based (33) and proposed control law respectively.

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Then, from the definition of u_ψ it follows that:

$$
\begin{aligned}
\int_0^\infty \left(x'_\psi Q x_\psi + u'_\psi R u_\psi \right) dt &\leq \int_0^T \left(x'_\psi Q x_\psi + u'_\psi R u_\psi \right) dt + x_\psi(T)' X^{-1}[\rho(T)] x_\psi(T) \\
&\leq \int_0^T \left(x'_{AMI} Q x_{AMI} + u'_{AMI} R u_{AMI} \right) dt + x_{AMI}(T)' X^{-1}[\rho(T)] x_{AMI}
\end{aligned}
\tag{36}
$$

From Schur complements, the set of inequalities (32) is equivalent to

$$
-\sum_{i=1}^{n_p} v_i \frac{\partial X}{\partial \rho} + A[\rho(t)] X(\rho) + X(\rho) A[\rho(t)]' + X(\rho) Q X(\rho) - B R^{-1} B' \leq 0
\tag{37}
$$

for all $\underline{v}_i \leq v_i \leq \overline{v}_i$ or, equivalently,

$$
-\sum_{i=1}^{n_p} v_i \frac{\partial X}{\partial \rho} + A_{cl} X + X \mathcal{A}'_{cl} + X Q X + B R^{-1} B' \leq 0
\tag{38}
$$

where $A_{cl}(\rho) = A[\rho(t)] - \frac{1}{2} B[\rho(t)] R^{-1} B[\rho(t)]' X(\rho)$ denotes the closed–loop dynamic matrix. Pre and post–multiplying this last equation by $x'_{AMI} X^{-1}$ and $X^{-1} x_{AMI}$ yields (after some algebra):

$$
\frac{d}{dt} \left(x'_{AMI} X^{-1} x_{AMI} \right) + x'_{AMI} Q x_{AMI} + u'_{AMI} R u_{AMI} \leq 0
\tag{39}
$$

Finally, integrating this last inequality yields

$$
\int_0^T \left(x'_{AMI} Q x_{AMI} + u'_{AMI} R u_{AMI} \right) dt + x_{AMI}(T)' X^{-1}[\rho(T)] x_{AMI} \leq x(0)' X^{-1}(\rho_o) x(0)
\tag{40}
$$

which, combined with (36) yields the desired result.

$\square$

6 Connections with other approaches

In this 'section we briefly explore the connections between the proposed controller and some related approached proposed in the past.

The basic idea of Algorithm 1, namely (i) to convert the infinite dimensional quadratic regulation problem to a finite dimensional optimization by using a penalty function to estimate the cost–to–go, and (ii) to use explicit optimization to drive the system to an invariant neighborhood of the origin and then switch controllers was proposed in [25, 26] for the case of constrained LTI systems (see also [5, 22] for later work along these lines).

The combination of dual mode control and receding horizon for stabilizing nonlinear systems was proposed in [18]. However, while this approach has the ability to robustly stabilize a nonlinear plant, does not address the issue of performance. Indeed, the performance index used there (and hence the control action) does not approach the optimal unless $T \to \infty$. From a practical stand–point, this implies that in order to

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achieve acceptable performance, the on–line optimization must be performed over a large horizon, which may not be feasible for moderately large plants or plants with fast time constants.

The extension of the techniques proposed in [25] to nonlinear plants was pursued in [16]. Here the infinite horizon cost is approximated by an expression of the form (11), where the terminal penalty function is obtained assuming that a stabilizing control law of the form $u = -Kx$ is available and will be used after the horizon T. Alternatively, the penalty function can be obtained by solving the Ricatti equation corresponding to the linearization of the dynamics around the equilibrium point $x = 0$ (see also [4]). Thus, this approach can be viewed as a special case of Algorithm 1 for a particular choice of (local) CLF.

The combination of Receding Horizon and CLF techniques has been proposed independently in [28, 29] and [20]. A different between these approaches is that the latter does not include a terminal penalty in the performance index. Rather, [20] optimizes a performance index of the form:

$$J = \int_{t}^{t+T} \left[x'Qx + u'Ru \right] d\tau \tag{41}$$

subject to

$$\begin{aligned} L_f V(x) + L_g V(x)u &\leq -\varepsilon\sigma(x) \\ V\left[x(t+T)\right] &\leq V\left[x_\sigma(t+T)\right] \end{aligned} \tag{42}$$

where $V(.)$ is a CLF and x_σ is the trajectory corresponding to the pointwise minimum norm control that renders $\dot{V} < 0$. This approach is guaranteed to stabilize the system for any horizon T such that the constraints (42) are feasible. However, it may do so at the expense of performance. Note that contrary to (11), (41) does not approximate the original cost, even if T is taken large or V coincides with the actual value function for the problem. Thus, the corresponding control action, while stabilizing, is not necessarily close to the optimal.

Finally, we close this section by showing that in the limit as $T \to 0$, the control law obtained from Algorithm 3 reduces to the inverse optimal controller proposed by Freeman and Kokotovic [8]. To this effect, note that if $T \to 0$ in (17) then u_Ψ is given by the solution to the following optimization problem:

$$u_\Psi = \underset{u}{\text{argmin}} \left\{ u'Ru + L_g \Psi u \right\} \tag{43}$$

subject to:

$$\psi_o + \psi_1 u = 0 \tag{44}$$

where:

$$\begin{aligned} \psi_o &= L_f \Psi[x(t)] + x(t)'Qx(t) + \sup_{\underline{v}_i \leq \dot{\rho}_i \leq \overline{v}_i} \sum_{i=1}^{n_\rho} \frac{\partial \Psi}{\partial \rho_i} \dot{\rho}_i + \alpha[x(t)] \\ \psi_1 &= L_g \Psi[x(t)] \end{aligned} \tag{45}$$

and where $-\alpha(x)$ is the desired negativity margin [8]. The solution to this optimization problem is given by:

$$u_\Psi = -\frac{\psi_o R^{-1} \psi_1'}{\psi_1 R^{-1} \psi_1'} \tag{46}$$

This is precisely the inverse optimal controller obtained in [8]

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7 Incorporating constraints

Next, we briefly discuss how to incorporate constraints into the formalism, proceeding in the spirit of [25, 27].

Assume for instance that the control action is constrained to belong to a given compact, convex, possibly parameter dependent set $u \in \mathcal{U}(\rho)$. Let V_u denote the *unconstrained* value function for the problem in a region S, and consider the set $\mathcal{U}_V = \left\{ x : u = -\frac{1}{2} g' \frac{\partial V_u}{\partial x} \in \mathcal{U}(\rho) \text{ for all } \rho \in \mathcal{P} \right\}$, i.e. the set of states where the unconstrained control law satisfies the control constraints. Finally, denote by S_V the largest invariant set contained in $\mathcal{U}_V$. Then, Algorithm 1 can be applied to the problem by simply modifying the explicit optimization (11) to take the constraints into account and using the set S_V as S_v. Clearly, this modified control law has the same properties as Algorithm 1, i.e. if the optimization is feasible, then it stabilizes the system and yields the optimal control action. Similarly, the use of a fixed optimization horizon T and an approximation $\Psi(x, \rho)$ can be taken into account by selecting Ψ to be a *constrained* control Lyapunov function in the sense that:

$$\min_{u \in \cap \mathcal{U}(\rho)} \sup_{\rho \in \mathcal{P}, \underline{v}_i \leq v_i \leq \overline{v}_i} \left\{ \frac{\partial \Psi}{\partial x} [f(x, \rho) + g(x, \rho)u] + \sum_i \frac{\partial \Psi}{\partial \rho_i} v_i \right\} \leq -\underline{\sigma} \|x\|^2 < 0; \, \forall x \in S_v \tag{47}$$

and modifying the algorithm so that the minimization is taken over $u \in \mathcal{U}(\rho)$ rather than over R^{n_u}. Clearly, this algorithm stabilizes the system in the region where this can be accomplished with bounded controls. Finally, state constraints can also be incorporated into the formalism by rendering suitable sets invariant, proceeding as in [27].

8 Illustrative Examples

This section illustrates our results with several examples. The first one is a simple academic example that can be solved analytically. Thus it can be used to explicitly analyze the source of performance degradation when using feedback linearization or the SDRE method, and to show the advantages of the proposed approach. The second example, a realistic problem arising in the context of control of thrust vectored aircrafts, was used for benchmarking several nonlinear design methods in [7]. Finally, the third example illustrates the advantages of the proposed method for the case of LPV systems.

Example 1: Consider the following regulation problem:

$$\min_u \left\{ J = \int_0^\infty x_2^2 + u^2 dt \right\} \tag{48}$$

subject to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 e^{x_1} + \frac{1}{2} x_2^2 + e^{x_1} u \end{aligned} \tag{49}$$

It can be shown that the optimal control law is given by:

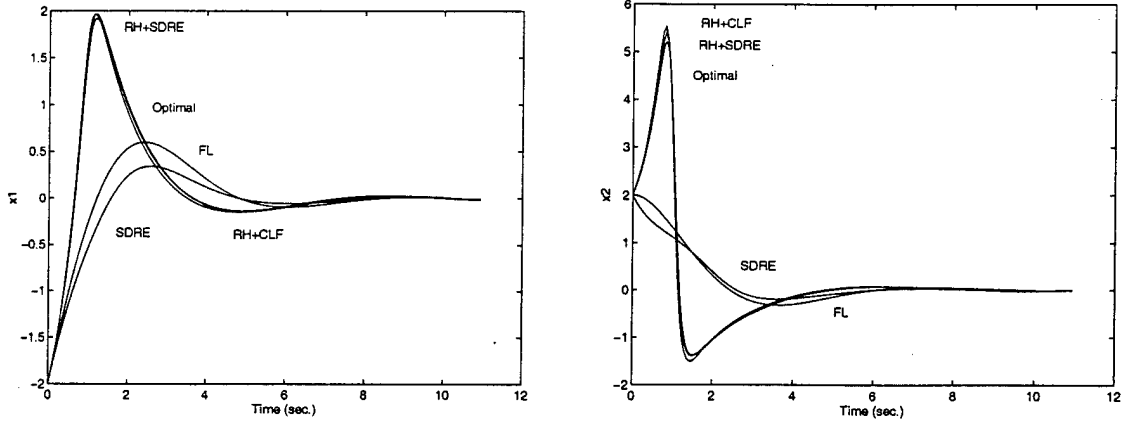$$u_{opt} = -x_2 \tag{50}$$

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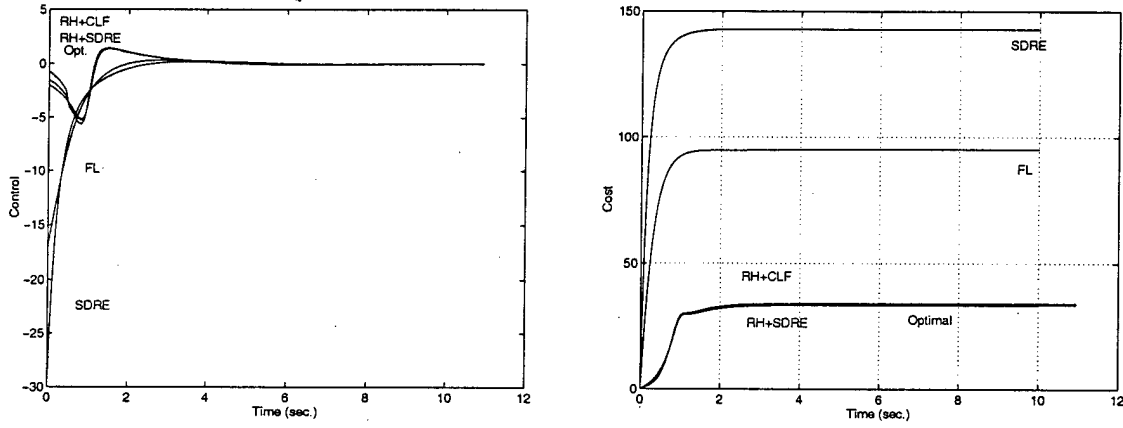Figure 2: State trajectories for example 1

Figure 3: Control effort and cost for example 1

with the corresponding optimal storage function:

$$V(x) = x_1^2 + x_2^2 e^{-x_1} \ . \tag{51}$$

A feedback linearization design selected so that the closed–loop system has the same storage function as $\|x\| \rightarrow 0$ yields the following controller and Lyapunov function:

$$
\begin{aligned}
u_{FL} &= \left(1 - e^{-x_1}\right)x_1 - x_2 e^{-x_1}\left(1 + 0.5 * x_2\right) \\
V_{FL} &= x_1^2 + x_2^2
\end{aligned} \tag{52}
$$

Note that $u_{FL} \cong u_{opt}$ only for small values of x_1 and x_2. Consider now the following state–dependent coefficient (SDC) parametrization:

$$A = \begin{bmatrix} 0 & 1 \\ -e^{x_1} & \frac{1}{2}x_2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ e^{x_1} \end{bmatrix} \tag{53}$$

It can be shown that the solution to the corresponding SDRE is given by:

$$P(x) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 1 \end{bmatrix} p(x) \tag{54}$$

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where

$$p(x) = \left(\frac{x_2}{2e^{x_1}} + \sqrt{1 + \left(\frac{x_2}{2e^{x_1}}\right)^2} \right) e^{-x_1} \tag{55}$$

with associated control action:

$$u_{sdre} = -x_2 \left[\left(\frac{x_2}{2e^{x_1}}\right) + \sqrt{1 + \left(\frac{x_2}{2e^{x_1}}\right)^2} \right] \tag{56}$$

Finally, it can also be shown that

$$x'P(x)x = pe^{x_1}V(x) \tag{57}$$

Thus $x'P(x)x$ gives a good estimate of $V(x)$ and $u_{sdre} \cong u_{opt}$ only when $\left(\frac{x_2}{2e^{x_1}}\right) \ll 1$.

Table 2 shows the different costs starting from the initial condition $x(0) = \begin{bmatrix} -2 & 2 \end{bmatrix}$ for several controllers, with the corresponding trajectories shown in Figures 2 and 3. The last two entries of the table correspond to the the proposed controller using an horizon $T = 1sec$ and as estimates of the value function $\Psi = V_{FL}$ and $\Psi = x'P(x)x$ respectively. Note that in this case performance of both controllers is virtually identical to optimal.

method	cost
optimal	33.56
FL	95.11
SDRE	143.0
RH+FL	34.24
RH+SDRE	33.7

Table 2: Comparison of different methods for Example 1

Example 2: Consider again the simplified model of the thrust vectored aircraft used in the introduction. Table 3 shows the cost corresponding to the initial condition $\xi(0) = \begin{bmatrix} 0 & 0 & 0 & 12.5 & 0 & 0 \end{bmatrix}$, obtained using different controllers. The two lowest entries correspond to the proposed method using $T = 1sec$ and $T_s = 0.5sec$ and terminal penalties derived from Jacobian Linearization and the SDRE methods respectively. Note that the latter virtually achieves optimal performance, while the former is only 2% suboptimal. This behavior can be explained by looking at Figure 4 that shows the different portions of the cost as a function of the horizon, starting from the initial condition $\xi(0)$. These plots show that while $\Psi(x) = x'P(x)x$ gives initially a very poor estimate of the cost–to–go, the combination of $\Psi(x)$ and the explicit integral in (17) give a very good estimate if T is chosen $\geq 1sec$. It is worth mentioning that a conventional receding horizon controller (i.e. one obtained by setting $\Psi \equiv 0$ in (17)) with the same choice of horizon and sampling time fails to stabilize the system.

Example 3: Consider an LPV system with the following state space realization:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0.5\rho - 1.5 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}',$$

$$C_1 = \sqrt{2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 & 1 \end{bmatrix}' \tag{58}$$

$$\mathcal{P} = \{\rho : 0 \leq \rho \leq 1\}; \underline{v} = -2, \bar{v} = 2$$

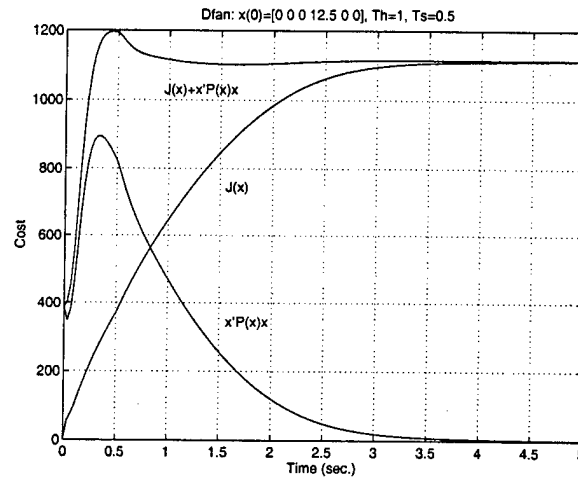| method	cost
exact	1115
LQR [7]	1.1×10^5
CLF [7]	2.53×10^4
LPV [7]	1833
SDRE	1640
RH+JL	1142 (T=1)
	1321 (T=0.4)
RH+SDRE	1117 (T=1)
	1310 (T=0.4)

Table 3: Comparison of different methods for Example 2

Figure 4: The terms of the cost as function of the horizon

It can be easily verified that the following matrix function satisfies the AMIs (32):

$$X(\rho) = X_o + X_1 \rho + X_2 \rho^2$$

$$X_o = \begin{bmatrix} 0.2210 & -0.3505 \\ -0.3505 & 1.1272 \end{bmatrix}, X_1 = \begin{bmatrix} -0.0239 & 0.0924 \\ 0.0924 & -0.3577 \end{bmatrix} \tag{59}$$

$$X_2 = \begin{bmatrix} 0.0243 & -0.0683 \\ -0.0683 & 0.2180 \end{bmatrix}$$

for all $\rho \in \mathcal{F}_v$. Figure 5[4] compares the trajectories starting from the initial condition $[0\ 2]'$ for the AMI–based $(x_{1lpv}, x_{2lpv}, u_{lpv})$ and the proposed controller respectively. The latter was implemented using $T = 2$ as horizon and $\Psi = x'X^{-1}(\rho)x$. For the specific parameter history shown there, the receding horizon controller yields $J = 6.91$ versus $J = 8.30$ for the control law (33), a performance improvement of roughly 20%. Similar results were obtained for other initial conditions and parameter trajectories.

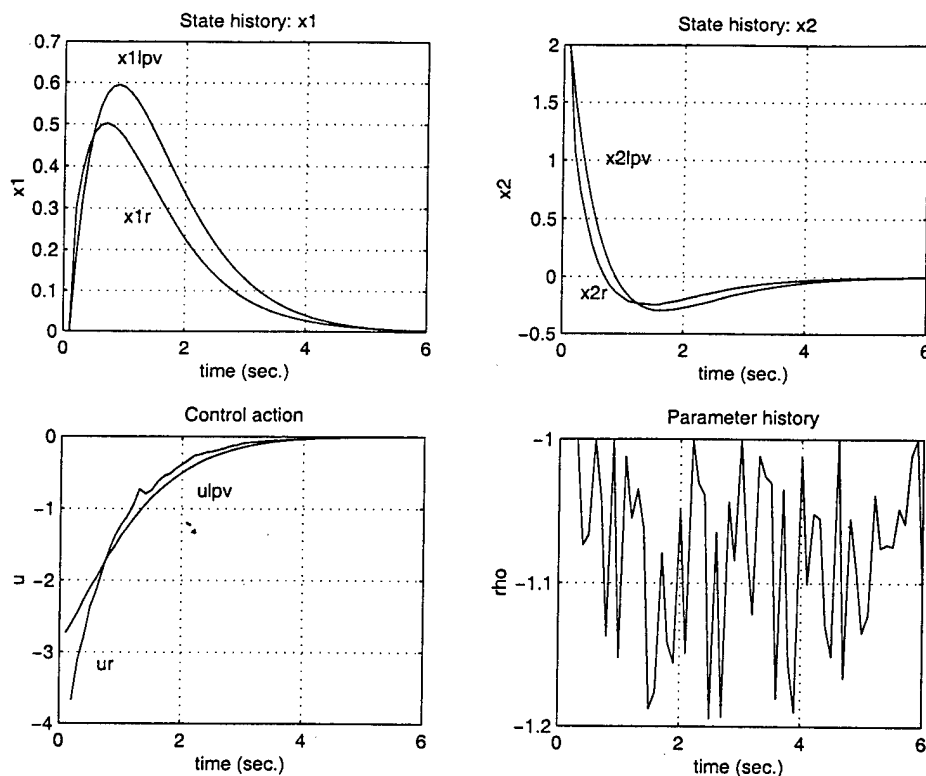[4]In this figure the parameter was normalized to $\rho_n = 0.5 * \rho - 1.5$.

Figure 5: state, control and (normalized) parameter trajectories for the Example

9 Conclusions

In contrast with the case of linear plants, tools for simultaneously addressing performance and stability of nonlinear systems have emerged relatively recently. Recent counterexamples [7] illustrated the fact that while several commonly used techniques can successfully stabilize nonlinear systems, the resulting closed–loop performance varies widely. Moreover, these performance differences are problem dependent, with performance of a given method ranging from (near) optimal to very poor.

In this chapter we have proposed a new suboptimal regulator for control affine parameter dependent non-linear systems, based upon the combination of receding horizon and control Lyapunov functions techniques. The main result of the chapter shows that under certain relatively mild conditions, essentially equivalent to the existence of a Control Lyapunov Function, this regulator renders the origin a globally asymptotically stable equilibrium point. Additional results show that for some readily available choices for the CLF $\psi(x, \rho)$ render the proposed controller near optimal for some cases of practical interest. These results were illustrated with a number of examples, where the proposed controller outperformed several other commonly used techniques.

An issue that was not addressed here is that of the computational complexity associated with solving the nonlinear optimization problem (17). Following [7] this complexity could be reduced by exploiting differential flatness to perform the optimization in flat space. Additional research being pursued includes the extension of the framework to the output feedback case and to handle model uncertainty.

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A   The SDRE approach to nonlinear regulation

In this section we briefly cover the details of the SDRE approach developed by Cloutier and coworkers [6, 19]. The main idea of the method is to recast the the nonlinear system (5) into a State Dependent Coefficient (SCD) linear–like form:

$$\dot{x} = A(x)x + B(x)u \qquad (60)$$

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and to solve pointwise along the trajectory the corresponding algebraic Riccati equation:

$$A'(x)P(x) + P(x)A(x) - P(x)B(x)R^{-1}(x)B'(x)P(x) + Q(x) = 0 \qquad (61)$$

The suboptimal control law is given by $u_{sdre} = -\frac{1}{2}R^{-1}(x)B'P(x)x$ where $P(x)$ is the positive definite (pointwise stabilizing) solution of (61). In the sequel we briefly review the properties of this control law. The corresponding proofs can be found in the appropriate references.

Lemma 4 ([6, 19]) *Assume that $Q(x) = C'(x)C(x)$ and that there exists a neighborhood Ω of the origin where the pairs $\{A(x), B(x)\}$ and $\{A(x), C(x)\}$ are pointwise stabilizable and detectable respectively and all the matrix functions involved are C^1. Then the control law u_{sdre} renders the origin a locally asymptotically stable equilibrium point of the closed–loop system.*

Lemma 5 ([6, 19]) *The SDRE control law and its associated state and co–state trajectories satisfy the following necessary condition for optimality: $\frac{\partial H}{\partial u} = 0$ where $H = x'Q(x)x + u'R(x)u + \lambda'[f(x) + B(x)u]$ is the Hamiltonian of the system and where λ denotes the co–states,*

Lemma 6 ([6, 19]) *Assume that the parametrization (60) is stabilizable and all the matrices involved along with their gradients are bounded in a neighborhood Ω of the origin. Then the SDRE control law and its associated state and co–state trajectories asymptotically satisfy at a quadratic rate[5] following necessary condition for optimality: $\dot{\lambda} = -\frac{\partial H}{\partial x}$ in the sense that*

$$\|\dot{\lambda} + \frac{\partial H}{\partial x}\| \leq x'Ux$$

for some constant matrix $U > 0$ and all $x \in \Omega$.

Lemma 7 ([12]) *Let $P(x)$ denote a solution to the SDRE (61). If there exists a positive definite function $V(x)$ such that $\frac{\partial V(x)}{\partial x} = P(x)x$ then u_{sdre} is the globally optimal control law[6].*

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[5]i.e. $\|\dot{\lambda} + \frac{\partial H}{\partial x}\| \to 0$ as $O(\|x\|^2)$ as $x \to 0$.

[6]A necessary and sufficient condition for this to hold is that the Jacobian matrix $\frac{\partial}{\partial x}[P(x)x]$ is symmetric (see for instance [13], section 3.1). In this case $V(x)$ can be computed as $V(x) = \int_o^x y'P(y) \cdot dy$, where this line integral is independent of the path.